STABILITY OF OPTIMAL SYSTEMS IN THE PRESENCE OF DISTURBANCES

(POMEKHOUSTOICHIVOST' OPTIMAL'NYKH SISTEM)

PMM Vol.27, No.3, 1963, pp. 459-467

V. A. PAL'MOV
(Leningrad)

(Received December 4, 1962)

The influence of disturbances on the performance of a time optimal linear system is considered when the control is subject to restricting conditions.

1. We shall consider an object, the state of which is determined by the elements of the column matrix $\xi(t)$, and the law of motion of which is given by the system of linear differential equations

$$\dot{\xi} = A\xi + Bu + CF \tag{1.1}$$

where A is a square matrix, and B and C are rectangular, with time-independent elements; $u(t)$, $F(t)$ are, respectively, column matrices of the control functions and external actions.

As in the book [1], we shall assume that the control region is a closed convex bounded polyhedron in the two-dimensional space with coordinates u^1, u^2, ..., u^r, where the u^r's are the elements of the control matrix.

The problem of finding the control $u(t)$ which moves the solution point from a position $\xi = \xi_0$ to the origin of the coordinates of the phase space ξ in a minimal time, is solved in the book [1]. It is shown that this problem can have solutions for some conditions imposed on F, only if A, B, C and F are known exactly.

We shall assume now, that the characteristics of the object (A, B, C), and the external forces F are random and only their statistical characteristics are known beforehand. In that case it is possible to construct an optimal control for the object having the expected characteristics and for external forces equal to their mathematical expectation. However, the realization of such a control for an object having random characteristics and subject to random external forces will not guarantee that the solution point will coincide with the origin of the coordinates at the end of the transient process.

The problem is yet complicated by the fact that random errors can be made in the determination of the optimal control.

The present work is concerned with the calculation of the dispersion of the phase coordinate at the end of the transient process and with the minimization of this dispersion by correcting the control.

2. We shall consider first a simplified problem, assuming that the parameters of the object, determined by the matrices A, B, C, are known exactly, and that the external forces, and in some respect the control, are random. We shall denote the mathematical expectation of the external force by Φ, and its mean random component by $\varphi(F = \Phi + \varphi)$.

We shall assume that the optimal control function $u = U$ which minimizes the duration of the transient process has been found when the object is subject to the expected external force Φ. The control is a piece-wise continuous function with discontinuities (changes of the control) at the points t_k ($k = 0, 1, 2, \ldots, n$). We shall denote by U_k the control matrix in the interval of time $t_k < t < t_{k+1}$. Let us denote by X the coordinates of the object during the transient process, and by X_k their values at the instant t_k.

In the interval of time

$$t_k < t < t_{k+1}$$

the state of the object is determined by the following equation and the initial condition

$$\dot{X} = AX + BU_k + C\Phi, \qquad X(t_k) = X_k \tag{2.1}$$

Its solution is the following, in matrix form [2]

$$X = e^{At}\left[e^{-At_k}X_k + \int_{t_k}^{t} e^{-A\mu}(BU_k + C\Phi)\,d\mu\right] \tag{2.2}$$

It is then possible to find the relation between two successive values of the coordinate at instants of switching of the control function

$$X_{k+1} = e^{At_{k+1}}\left[e^{-At_k}X_k + \int_{t_k}^{t_{k+1}} e^{-A\mu}(BU_k + C\Phi)\,d\mu\right] \tag{2.3}$$

We shall call unperturbed the optimal transient process considered.

Let us now search for the perturbed transient process of the object (1.1). We shall assume that the control in that process has the same number of discontinuities and takes the same values as in the case of the unperturbed process, however the switching occurs at instants $t_k + \tau_k$, where the τ_k's are random values.

Thus, in the interval of time

$$t_k + \tau_k < t < t_{k+1} + \tau_{k+1} \tag{2.4}$$

the control has the values U_k, and the relation between the coordinates at the end and the beginning of the interval (2.4) is found by integration of equation (1.1)

$$\xi_{k+1} = \exp\left[A\left(t_{k+1} + \tau_{k+1}\right)\right]\Big[\exp\left[-A\left(t_k + \tau_k\right)\right]\xi_k + \\ + \int_{t_k+\tau_k}^{t_{k+1}+\tau_{k+1}} e^{-A\mu}\left(BU_k + CF\right)d\mu\Big] \tag{2.5}$$

We shall consider the disturbances in the coordinates of the object $x_k = \xi_k - X_k$ at the instants of switching. Assuming that x_k, τ_k, φ, are small values, we shall linearize the relation (2.5). Taking (2.3) and (2.1) into consideration, we get as a result of the linearization

$$x_{k+1} = \exp\left[A\left(t_{k+1} - t_k\right)\right]x_k + \int_{t_k}^{t_{k+1}} \exp\left[A\left(t_{k+1} - \mu\right)\right]C\varphi(\mu)\,d\mu + \\ + \dot{X}_{k+1}^{-}\tau_{k+1} - \exp\left[A\left(t_{k+1} - t_k\right)\right]\dot{X}_k^{+}\,\tau_k \tag{2.6}$$

where $k = 0, 1, 2, \ldots, n-1$, and $\dot{X}_k^-$ and $\dot{X}_k^+$ are the values of the velocity for the optimal transient process before and after the switching at the instant t_k. We shall note that the values of the velocity before the first switching ($\dot{X}_0^-$) and after the final one ($\dot{X}_n^+$) do not enter equation (2.6).

We shall consider the first variant of the realization of an optimal control, when it is given in the function of time. Here the switchings occur at the instants $t_k + \tau_k$, where the errors τ_k on the instants of switching are independent of x. Then (2.6) represents a linear difference equation with respect to x_k, and its solution can be constructed by a method analogous to that exposed in the book [3]. The perturbation x_n at the end of the transient process is equal to

$$x_n = \exp\left[A\left(t_n - t_0\right)\right]x_0 + \int_{t_0}^{t_n} \exp\left[A\left(t_n - \mu\right)\right]C\varphi(\mu)\,d\mu + \dot{X}_n^{-}\tau_n +$$

$$+\sum_{k=1}^{n-1}\exp\left[A(t_n-t_k)\right](\dot{X}_k^- - \dot{X}_k^+)\,\tau_k - \exp\left[A(t_n-t_0)\right]\dot{X}_0^+\tau_0 \quad (2.7)$$

Equation (2.7) can be rewritten in a more compact form

$$x_n = \exp\left[A(t_n-t_0)\right]x_0 + \int_{t_0}^{t_n}\exp\left[A(t_n-\mu)\right]C\varphi d\mu +$$

$$+\sum_{k=0}^{n}\exp\left[A(t_n-t_k)\right]\Delta_k\tau_k \quad (2.8)$$

if we denote

$$\Delta_k = \dot{X}_k^- - \dot{X}_k^+ \quad (2.9)$$

and assume that

$$\dot{X}_n^+ = 0, \qquad \dot{X}_0^- = 0 \quad (2.10)$$

We shall assume that the perturbations τ_k, x_0, φ are independent of one another, and have a mathematical expectation equal to zero. Then $M[x_n] = 0$ and the matrix of the correlation instants [4] composing the column x_n has the form

$$D_n^x = \exp\left[A(t_n-t_0)\right]D_0^x\exp\left[A'(t_n-t_0)\right] +$$

$$+\int_{t_0}^{t_n}\int_{t_0}^{t_n}\exp\left[A(t_n-\mu)\right]CK_\varphi(\mu,\nu)\,C'\exp\left[A'(t_n-\nu)\right]d\nu d\mu +$$

$$+\sum_{k=0}^{n}\exp\left[A(t_n-t_k)\right]\Delta_k\Delta_k'\exp\left[A'(t_n-t_k)\right]D_k^\tau \quad (2.11)$$

Here $M[\ldots]$ is the operation of mathematical expectation; $K\varphi(\mu, \nu)$ is the matrix of the correlation function composing φ

$$K_\varphi(\mu,\nu) = M\left[\varphi(\mu)\varphi'(\nu)\right] \quad (2.12)$$

$D_k^\tau = M[\tau_k^2]$ are the standard deviations of the instants of switching of the controls; A' is the transposed matrix of A, etc.

The trace of the correlation matrix D_n^x can be used as a measure of the dispersion of x_n.

3. We shall consider another variant of the realization of an optimal control, when it is not given as a function of time as above, but as a function of the successive values of the phase coordinates of the object. The operation of the optimal system in the undisturbed transient process is the following. A measurement of the phase coordinates X is made, and when the condition $X = X_k$ is satisfied the control switches from the value U_{k-1} to the value U_k. The corresponding phase trajectory is shown by the thick line on Fig. 1. In the presence of perturbations φ the switching should not be made on points of the phase space, but on some switching hypersurface

$$\psi_k(\xi_k) = 0 \qquad (k = 1, 2, \ldots, n)$$

These surfaces are also represented in Fig. 1. The light lines represent the phase trajectories corresponding to different random realizations of the perturbation $\varphi(t)$.

In order to take into account the inaccuracies of the measurement of the phase coordinates of the object, we shall assume that the moments of the controls are determined by the equation

$$\psi_k(\xi_k - y_k) = 0 \qquad (k = 1, 2, \ldots, n) \tag{3.1}$$

Here the measurement errors y_k are random values, thus the equation (3.4) determines a family of surfaces on which the kth switching occurs. Then the value U_{k-1} of the control is replaced by the value U_k.

Taking into consideration that $\xi_k = X_k + x_k$ and assuming that the perturbations x_k and the errors y_k are small, we shall find an approximate equation of the switching hypersurface

$$\psi_k(X_k) + m_k(x_k - y_k) = 0 \tag{3.2}$$

Here m_k is a row matrix, consisting of the derivatives with respect to ξ's for $\xi = X_k$ of the scalar function $\psi_k(\xi)$.

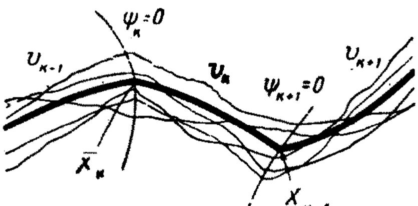

Fig. 1.

In the absence of perturbations, $y_k = 0$, $x_k = 0$, and the switchings occur for $\xi = X_k$.

There follows from (3.2)

$$\psi_k(X_k) = 0 \tag{3.3}$$

and the equation of the switching hypersurfaces takes the simple form

$$m_k x_k = m_k y_k \tag{3.4}$$

This is the equation of the hyperplane tangent to the hypersurface (3.1). By virtue of the randomness of y_k, there is a family of parallel switching hyperplanes (3.4). The matrix m_k determines their common normal vector, whereupon its components are proportional to the corresponding elements of m_k. We shall note that $\psi_k(\xi)$ are as yet arbitrary functions depending upon the condition (3.3). There follows that the elements of the matrices m_k are arbitrary numbers, and it can be hoped that by making an appropriate choice, it would be possible to reduce the dispersion of the phase coordinates at the end of the transient process.

First we shall find the statistical properties of this dispersion. The relation (2.6) is still valid in the present case, although the values τ_k are not independent any more, and must be determined from the equation of the switching surfaces (3.4). For further use, it is convenient to write the equation (2.6) in the following form:

$$x_{k+1} = Q_k\,(x_k - \dot{X}_k^{+}\tau_k) + \int_{t_0}^{t_n} P_k\,(\mu)\,C\varphi\,(\mu)\,d\mu + \dot{X}_{k+1}^{-}\tau_{k+1} \tag{3.5}$$

Here the notations are the following

$$Q_k = \exp\,[A\,(t_{k+1} - t_k)]$$

$$P_k\,(\mu) = \begin{cases} 0 & (t_0 \leqslant \mu \leqslant t_k) \\ \exp\,[A\,(t_{k+1} - \mu)] & (t_k < \mu \leqslant t_{k+1}) \\ 0 & (t_{k+1} < \mu \leqslant t_n) \end{cases} \tag{3.6}$$

Substituting the expression of x_{k+1} given by the equation (3.5) into the equation of the switching surface (3.4) we find τ_{k+1}

$$\tau_{k+1} = \frac{m_{k+1}}{m_{k+1}\dot{X}_k^{-}}\Big[\,y_k - Q_k\,(x_k - \dot{X}_k^{+}\tau_k) - \int_{t_0}^{t_n} P_k\,C\varphi d\mu\,\Big] \tag{3.7}$$

Using the equations (3.5) and (3.7) we obtain the expression

$$x_{k+1} - \dot{X}_{k+1}^{+}\tau_{k+1} = \gamma_k Q_k\,(x_k - \dot{X}_k^{+}\tau_k) + \gamma_k \int_{t_0}^{t_n} P_k C\varphi d\mu + \rho_k y_{k+1} \tag{3.8}$$

Here we have introduced the notation

$$\rho_k = \frac{\Delta_{k+1} m_{k+1}}{m_{k+1}\dot{X}_{k+1}^{-}}\,, \qquad \gamma_k = E - \rho_k \tag{3.9}$$

whereby E is the unit matrix of appropriate order, and Δ_k is determined by the equation (2.9). The numerator of the expression for ρ_k is a square matrix and the denominator a number. This follows from the fact that Δ_{k+1} is a column matrix and m_{k+1} a row matrix with the same number of components.

The equation (3.7) is a linear difference equation with respect to $x_k - \dot{X}_k^{+}\tau_k$ and its solution can be found as previously. In as much as the condition (2.10) is satisfied, the expression $x_n - \dot{X}_n^{+}\tau_n$ coincides with the unknown value x_n

$$x_n = \int_{t_0}^{t_n} S(\mu)\, C\varphi(\mu)\, d\mu + \sum_{k=1}^{n} T_k y_k + \Pi\,(x_0 + \Delta_0\tau_0) \tag{3.10}$$

Here the notation is the following:

$$S(\mu) = \sum_{k=0}^{n-1} \Upsilon_{n-1}Q_{n-1}\Upsilon_{n-2}Q_{n-2}, \ldots, \Upsilon_{k+1}Q_{k+1}\Upsilon_k P_k(\mu) \tag{3.11}$$

$$T_k = \Upsilon_{n-1}Q_{n-1}, \ldots, \Upsilon_k Q_k \rho_{k-1}$$

$$\Pi = \Upsilon_{n-1}Q_{n-1}, \ldots, \Upsilon_0 Q_0$$

We shall assume that the values y_k, x_0, τ_0 are statistically independent of one another and have a mathematical expectation equal to zero. The probability characteristics of x_n are found by using equation (3.10)

$$M\,[x_n] = 0$$

$$D_n^{x} = \int_{t_0}^{t_n}\int_{t_0}^{t_n} S(\mu)\, CK_{\varphi}(\mu, \nu)\, C'S'(\nu)\, d\nu d\mu + \sum_{k=1}^{n} T_k D_k^{y} T_k' + \Pi\,(D_0^{x} + \Delta_0\Delta_0' D_0^{\tau})\,\Pi' \tag{3.12}$$

where D_k^{y} is the correlation matrix of the measurement errors, and D_0^{x} of the errors on the initial conditions; D_0^{τ} is the standard deviation τ_0.

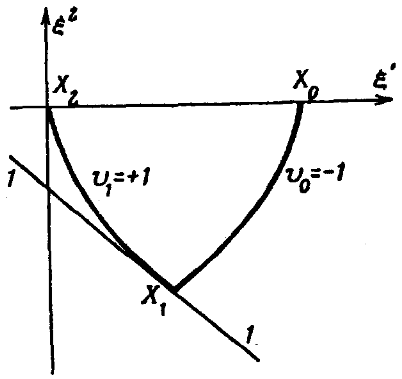

Fig. 2.

As a measure of the dispersion of the coordinates at the end of the transient process, it is possible to take the trace of the matrix D_n^{x} or any other quantity composed of its elements. We shall denote by I the chosen measure. It appears as a complicated function of the row matrices m_k which compose it. In order to reduce the dispersion of x_n it is possible to choose an m_k such that the quantity I has its minimal value.

We shall call optimal the position of the switching hyperplane (3.4) corresponding to these values of m_k.

4. Let us consider an example. Let the transient process of the object be defined by the following equation and initial conditions:

$$\ddot{z} = u + \varphi, \qquad z_0 > 0, \qquad \dot{z}_0 = 0$$

The domain of control is $|u| \leqslant 1$.

We shall write the equations and the initial conditions in the standard form

$$\dot{\xi} = A\xi + Bu + C\varphi \tag{4.1}$$

where

$$\xi = \left\| \begin{matrix} z \\ \dot{z} \end{matrix} \right\|, \quad A = \left\| \begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix} \right\|, \quad B = C = \left\| \begin{matrix} 0 \\ 1 \end{matrix} \right\|, \quad \xi_0 = \left\| \begin{matrix} z_0 \\ 0 \end{matrix} \right\| \tag{4.2}$$

The solution of the problem for $\varphi = 0$ is given in the book [1]. The corresponding phase trajectory is given in Fig. 2. It consists of two sections of parabolas. The transient process lasts a time $T = 2\sqrt{z_0}$ where $u = U_0 = -1$ for the first half of the time and $u = U_1 = 1$ for the second.

We shall note that the equation of the tangent I (Fig. 2) to the second parabola at the point of switching of the control has the form:

$$x^2 + \frac{2}{T} x' = 0 \tag{4.3}$$

Here the indices above the letters denote the number of the component of the corresponding column. Equation (4.3) is the equation of the tangent to the "switching curve" at the switching point for the undisturbed optimal process. The notion of "switching curve" in the present case is the same as in the book [1].

Considering now the influence of the disturbances on this transient process, we shall assume that the errors on the switching instants in the first variant of realization of optimal control and the errors on the measurement of the phase coordinates in the second are equal to zero, i.e. $D^{\tau} = 0$, $D_k^{\ y} = 0$, $D_0^{\ x} = 0$. By virtue of the special form of the matrix A, we have

$$e^{At} = E + At = \left\| \begin{matrix} 1 & t \\ 0 & 1 \end{matrix} \right\| \tag{4.4}$$

Therefore the trace of the matrix of the correlation moments of x_n, determined by the equation (2.11) has the following expression:

$$I_1 = \operatorname{Sp} D_2^x = \int_0^T \int_0^T [(T - \mu)(T - \nu) + 1] K_\varphi(\mu, \nu)\, d\nu\, d\mu \tag{4.5}$$

If the control is a function of the phase coordinates of the object, then the dispersion of the coordinates at the end of the transient process is determined by the formula (3.12). The values entering this equation have the following form

$$\dot{X}_1^- = -\begin{Vmatrix} T/2 \\ 1 \end{Vmatrix}, \qquad \dot{X}_1^+ = -\begin{Vmatrix} T/2 \\ -1 \end{Vmatrix}, \qquad \dot{X}_2^- = \begin{Vmatrix} 0 \\ 1 \end{Vmatrix}, \qquad \dot{X}_2^+ = 0$$

$$P_0(\mu) = \begin{cases} \begin{Vmatrix} 1 & T/2 - \mu \\ 0 & 1 \end{Vmatrix} & \left(0 \leqslant \mu \leqslant \frac{T}{2}\right), \\ 0 & \left(\frac{T}{2} < \mu \leqslant T\right), \end{cases} \qquad P_1(\mu) = \begin{cases} 0 & \left(0 \leqslant \mu \leqslant \frac{T}{2}\right) \\ \begin{Vmatrix} 1 & T - \mu \\ 0 & 1 \end{Vmatrix} & \left(\frac{T}{2} < \mu \leqslant T\right) \end{cases}$$

$$\gamma_0 = \begin{Vmatrix} 1 & 0 \\ \dfrac{-2\alpha_1}{1 + \alpha_1 T/2} & \dfrac{\alpha_1 T/2 - 1}{1 + \alpha_1 T/2} \end{Vmatrix}, \qquad \gamma_1 = \begin{Vmatrix} 1 & 0 \\ -\alpha_2 & 0 \end{Vmatrix}, \qquad Q_1 = \begin{Vmatrix} 1 & T/2 \\ 0 & 1 \end{Vmatrix}$$

Here we have introduced the notation

$$\alpha_1 = \frac{m_1^1}{m_1^2}, \qquad \alpha_2 = \frac{m_2^1}{m_2^2}$$

where m^1, m^2 are components of the m matrices.

We shall substitute the required expressions into (3.11), (3.12) and we shall compute the trace of the matrix of the correlation moments of x_n

$$I = \operatorname{sp} D_2^x = (1 + \alpha_2^2)\left[\left(\frac{1 - \alpha_1 T/2}{1 + \alpha_1 T/2}\right)^2 M_1 - 2\,\frac{1 - \alpha_1 T/2}{1 + \alpha_1 T/2}\, M_2 + M_3\right] \tag{4.6}$$

The notation is the following:

$$M_1 = \int_0^{T/2} \int_0^{T/2} \mu\nu K_\varphi(\mu, \nu)\, d\nu d\mu, \qquad M_2 = \int_0^{T/2} \int_{T/2}^{T} \mu (T - \nu) K_\varphi(\mu, \nu)\, d\nu d\mu$$

$$M_3 = \int_{T/2}^{T} \int_{T/2}^{T} (T - \mu)(T - \nu) K_\varphi(\mu, \nu)\, d\nu d\mu \tag{4.7}$$

The quantity (4.6) reaches its minimal value

$$I_2 = M_3 - \frac{(M_2)^2}{M_1} \tag{4.8}$$

for the following values of α_1 and α_2:

$$\alpha_1^* = \frac{2}{T}\,\frac{1 - M_2 / M_1}{1 + M_2 / M_1},\ \alpha_2^* = 0 \tag{4.9}$$

If we assume $\alpha_2 = 0$, and if the curve (4.3) ($\alpha_1 = 2/T$) is taken as the curve of the first switching, the dispersion of the coordinates is determined by the value

$$I_3 = M_3 \tag{4.10}$$

We shall compare the quantities I_1, I_2 and I_3 for two aspects of random perturbations: in the case of a small and of a large correlation time. If the correlation time is much smaller than T, we can write for the calculation

$$K_\varphi(\mu,\ \nu) = H\delta(\mu - \nu)$$

where δ is the delta-function, and H = constant.

The computation gives

$$\alpha_1^* = \frac{2}{T}, \qquad I_1 = H\left(\frac{T^3}{3} + T\right), \qquad I_2 = \frac{HT^3}{24}, \qquad I_3 = \frac{HT^3}{24}$$

For a disturbance of the type considered, the optimal position of the switching curve for which I is minimal, coincides with line *1*. Random phase trajectories for that case are shown in Fig. 3. If the correlation

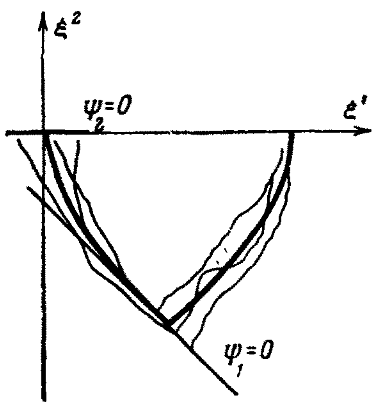

Fig. 3.

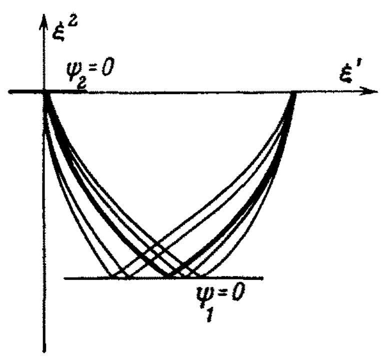

Fig. 4.

time of the random process φ is much larger than the time of the undisturbed transient process T, then we can take approximately

$$K_\varphi(\mu,\ \nu) = R = \text{const}$$

The corresponding calculation yields

$$\alpha_1^* = 0, \qquad I_1 = R\left(\frac{T^4}{4} + T^2\right), \qquad I_2 = 0, \qquad I_3 = \frac{RT^4}{64}$$

The optimal position of the switching curve, when the object is subject to the action of disturbances with a large correlation time is shown in Fig. 4.

5. We shall give, without derivation, the equations analogous to (2.11) and (3.12), when the characteristics of the object are also random and are determined by the matrices $A + a$, $B + b$ and $C + c$, where a, b, c are centered random components which we shall consider small.

If the optimal control is realized in the function of time, then the correlation matrix of the moments of x_n takes the form

$$D_n^x = \exp\,[A\,(t_n - t_0)]\,D_0^x \exp\,[A'\,(t_n - t_0)] + \sum_{k=0}^{n} \exp\,[A\,(t_n - t_0)]\,\Delta_k \times$$
$$\times\,\Delta_k' \exp\,[A'\,(t_n - t_k)]\,D_k^\tau + \int_{t_0}^{t_n}\int_{t_0}^{t_n} \exp\,[A\,(t_n - \mu)]\,[CK_\varphi\,(\mu, \nu)\,C' +$$
$$+ N\,(\mu, \nu)]\exp\,[A'\,(t_n - \nu)]\,d\nu\,d\mu \qquad (5.1)$$

When the optimal control is realized in the function of the phase coordinates of the object, the correlation matrix of the dispersion is the following:

$$D_n^x = \sum_{k=1}^{n} T_k D_k^y T_k' + \Pi\,(D_0^x + \Delta_0\Delta_0' D_0^\tau)\,\Pi' +$$
$$+ \int_{t_0}^{t_n}\int_{t_0}^{t_n} S\,(\mu)\,[CK_\varphi\,(\mu, \nu)\,C' + N\,(\mu, \nu)]\,S'\,(\nu)\,d\nu\,d\mu \qquad (5.2)$$

In equations (5.1) and (5.2) the matrix N has the following meaning:

$$N(\mu,\nu) = M\,[\{aX(\mu) + bU(\mu) + c\Phi(\mu)\}\,\{X'(\nu)\,a' + U'(\nu)\,b' + \Phi'(\nu)\,c'\}] \qquad (5.3)$$

The remaining notation is the same as previously.

Let us assume for instance that the behavior of the object is described by the equation

$$\ddot{z} = (1 + \varepsilon)\,u \qquad (5.4)$$

where ε is a small random quantity with a zero average value. We shall take the initial conditions and the limitations for the control identical as those of Section 4. Then the undisturbed transient process (for $\varepsilon = 0$) and the parameters of the equation of the object in standard form will be the same as in Section 4 and the matrix b takes the form

$$b = C\varepsilon \tag{5.5}$$

The comparison of the corresponding equations shows that the solution of the problem of the dispersion at the end of the transient process in the present case is given by the equations (4.5), (4.6), (4.7) and (4.9) if $K_{\varphi}(\mu, \nu)$ is replaced in these equations by $D^{\varepsilon}U(\mu)U(\nu)$ where D^{ε} is the standard deviation ε.

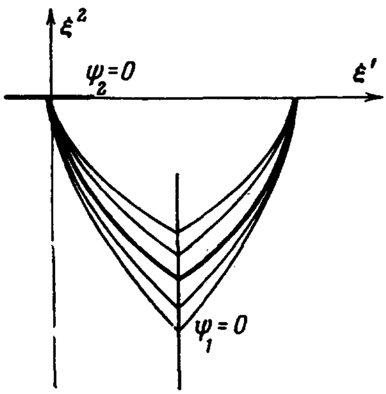

Fig. 5.

The results of the computation are the following:

$$\alpha_1^* = \infty, \quad I_1 = D^{\varepsilon}\frac{T^4}{16}$$
$$I_2 = 0, \quad I_3 = D_{\varepsilon}\frac{T^4}{64}$$

The optimal position of the switching surfaces and the random phase trajectories are shown in Fig. 5.

BIBLIOGRAPHY

1. Pontriagin, L.S., Boltianskii, V.G., Gamkrelidze, R.V. and Mishchenko, E.F., *Matematicheskaia teoriia optimal'nykh protsessov* (English translation - *The Mathematical Theory of Optimal Processes*, John Wiley & Sons). Fizmatgiz, 1961.

2. Gantmakher, F.R., *Teoriia matrits (Matrix Theory)*. Gostekhizdat, 1954.

3. Gel'fond, A.O., *Ischislenie konechnykh raznostei (Calculus of finite differences)*. Gostekhizdat, 1952.

4. Pugachev, V.S., *Teoriia sluchainykh funktsii (Theory of random functions)*. Fizmatgiz, 1960.

Translated by A.V.